Computing the Shapley value to share ressources in cooperative games

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Context

RETAIL
RESPONSABLE

- Focus
  - Study a more environmentally/socially friendly online retail

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- Industrial partner : One Stock

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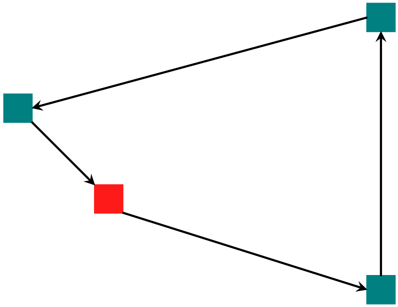
  - Study a more environmentally/socially friendly online retail
- Industrial partner : One Stock
- Main participants
  - Sandra U. Ngueveu (Holder)
  - Francois Lamothe (Post-doc)
  - Théo Le Brun (PhD Student)
- Open to new collaborators

Cost sharing indicator

Package delivery system

- Service to a set of customers $\rightarrow$ cost
- Economies of scale



Vehicle routing problem

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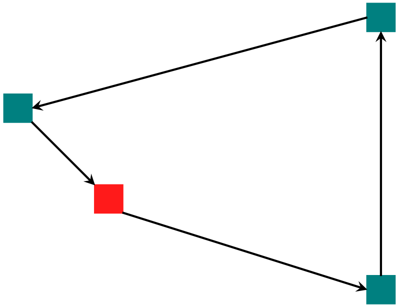

Question :

How to divide the total cost amongst the users ?

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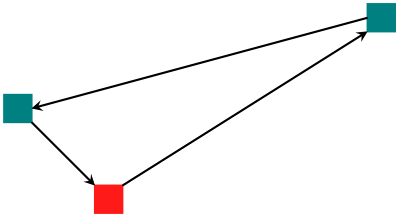

Dividing equally among the customers ?

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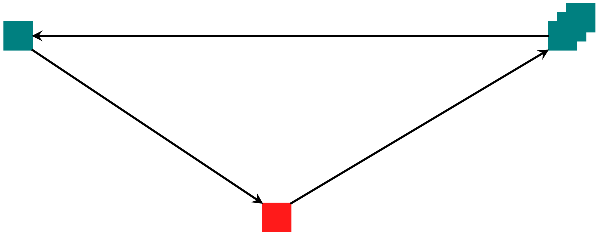

Dividing dividing proportionally to the distance to the depot ?

Cooperative games and Shapley value : definition

Cooperative game

- A set of player N : customers

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  - v(S) : Cost of servicing the players in S

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  - Linearity : works well with sum of games

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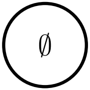

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Shapley value :

$$\emptyset \xrightarrow{\Delta(S_1^{\pi}, 1)} \{1\}$$

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$$\begin{tikzcd}[column sep=large] \emptyset \arrow[r, "\Delta(S_1^\pi, 1)"] & \{1\} \arrow[r, "\Delta(S_1^\pi, 2)"] & \{1, 2\} \arrow[r, dashed] & N \end{tikzcd}$$

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Shapley value :

$$\emptyset \xrightarrow{\Delta(S_1^\pi, 1)} \{1\} \xrightarrow{\Delta(S_1^\pi, 2)} \{1, 2\} \dashrightarrow N$$

- Average over all possible orders :
  - $\phi(v, i) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \Delta(S_i^\pi, i)$
- NP-Hard to compute $\to$ approximation

Approximation with sampling

Shapley value : $\phi(v, i) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \Delta(S_i^\pi, i)$

Shapley value as an expectation : $\phi(v, i) = \mathbb{E}_\pi[\Delta(S_i^\pi, i)]$

Approximation with sampling

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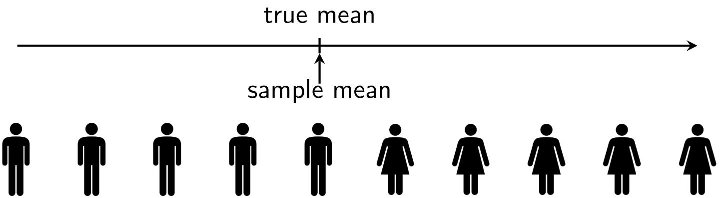
Classical approximation method :

- Expectation $\rightarrow$ average over samples
- Sampling a set of permutations Π
- $\phi(v, i) \approx \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \Delta(S_i^\pi, i)$

Stratified sampling

Stratified sampling

Example : estimating mean human size



Stratified sampling

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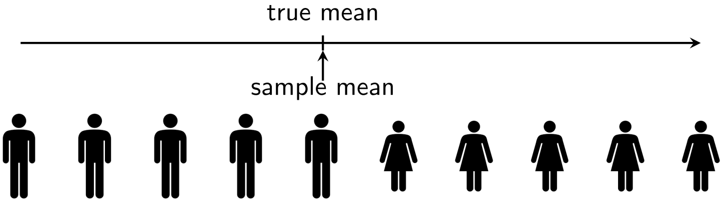

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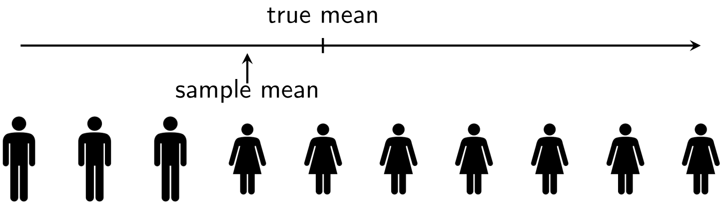

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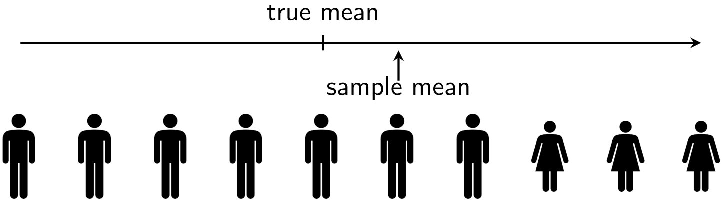

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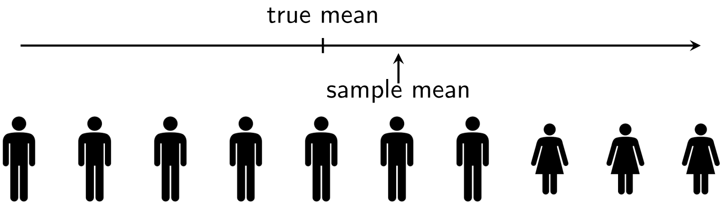

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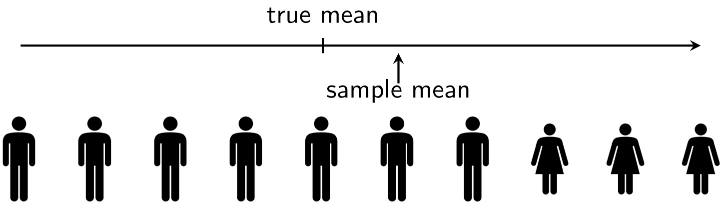

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- Example : mean size = (mean man size + mean woman size) /2

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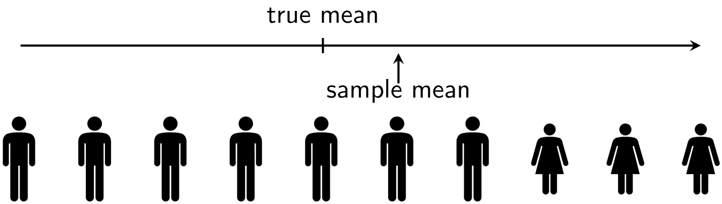

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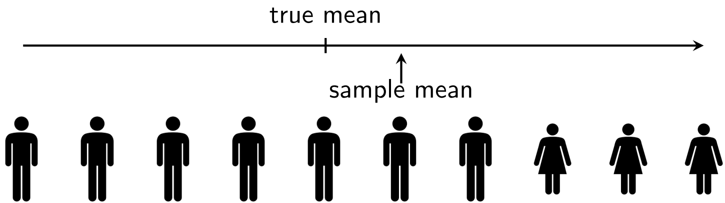

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Law of total expectations : $\mathbb{E}[\hat{\theta}] = P(A)\ \mathbb{E}[\hat{\theta}|A] + P(\bar{A})\ \mathbb{E}[\hat{\theta}|\bar{A}]$

Stratified estimator : $\frac{P(A)}{|\Theta_A|} \sum_{\theta \in \Theta_A} \theta + \frac{P(\bar{A})}{|\Theta_{\bar{A}}|} \sum_{\theta \in \Theta_{\bar{A}}} \theta$

Stratification events for the Shapley value

Shapley value : $\phi(v, i) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \Delta(S_i^\pi, i)$

Groups :

- What is the position of player i in permutation π ? $\rightarrow$ |N| groups
- Is player j before or after player i ? $\rightarrow$ 2 groups

How to use and combine all this groups/stratifications ?

Flexible stratification with optimization

Problem : we can't stratify according to 2 events independently.

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- Estimate $\mathbb{E}[\theta|A \cap B]$ and $\mathbb{E}[\theta|A \cap \bar{B}]$ and $\mathbb{E}[\theta|\bar{A} \cap B]$ and $\mathbb{E}[\theta|\bar{A} \cap \bar{B}]$ $\rightarrow 2^n$ estimates

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$\rightarrow 2^n$ estimates

- Estimate $\mathbb{E}[\theta|A]$ and $\mathbb{E}[\theta|\bar{A}]$ and $\mathbb{E}[\theta|B]$ and $\mathbb{E}[\theta|\bar{B}]$
$\rightarrow$ How to combine them ?

Flexible stratification with optimization

New paradigm :

- We are estimating the expectation $\mathbb{E}[\theta]$ with a weighted average of sample
- Each sample σ has a weight w_σ
- If we know the proba $P(A)$ of event A : we want $\sum_{\sigma \in \Sigma_A} w_\sigma = P(A)$

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We obtain a linear system of equations :

$$W(A) = \sum_{\sigma \in \Sigma_A} w_\sigma = P(A) \quad \forall A$$

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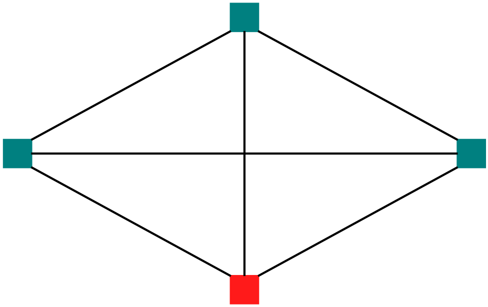
Modeling with a QP :

$$\min_{w_\sigma} \sum_A (W(A) - P(A))^2 + \epsilon \sum_\sigma (w_\sigma)^2$$

$$s.t. \sum_\sigma w_\sigma = 1$$

Shapley value 😇 vs core 😈

Example : all length $= 1$; vehicule capacity $= 2$



Shapley value 😇 vs core 😈

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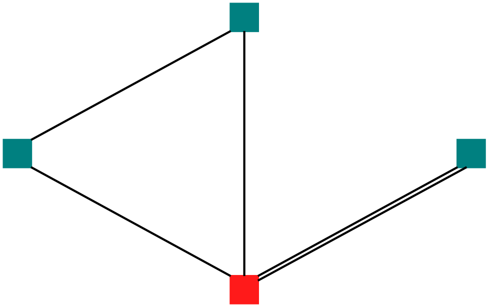

- Best value $= 5$; Shapley value $= 5/3$

Shapley value 😇 vs core 😈

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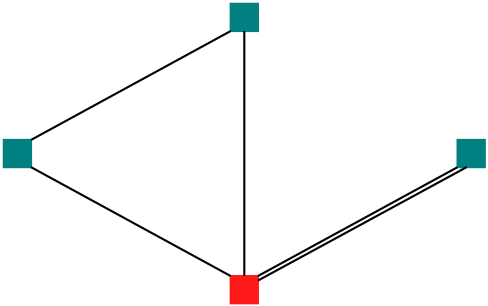

- Best value $= 5$; Shapley value $= 5/3$
- Best 2 size coalition : 3 ; Shapley value $= 3/2$

Shapley value 😇 vs core 😈

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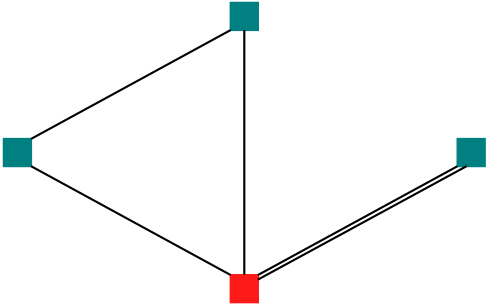

- Best value = 5 ; Shapley value = 5/3
- Best 2 size coalition : 3 ; Shapley value = 3/2
- Core can be non-empty but not contain the Shapley value

Thank you for your attention

Package
https://github.com/TwistedNerves/shapley_approximation

Paper :
Approximating the Shapley value with sampling : survey and new stratification techniques, Francois Lamothe, Sandra U. Ngueveu